Variational study of bound states in the Higgs model

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The possible existence of Higgs-boson–Higgs-boson bound states in the Higgs sector of the standard model is explored using the $|hh\rangle + |hhh\rangle$ variational ansatz of Di Leo and Darewych. The resulting integral equations can be decoupled exactly, yielding a one-dimensional integral equation, solved numerically. We thereby avoid the extra approximations employed by Di Leo and Darewych, and we find a qualitatively different mass renormalization. Within the conventional scenario, where a not-too-large cutoff is invoked to avoid ''triviality,'' we find, as usual, an upper bound on the Higgs boson mass. Bound-state solutions are only found in the very strong coupling regime, but at the same time a relatively small physical mass is required as a consequence of renormalization.

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I. INTRODUCTION

The possible existence of bound states for the Higgs boson has been studied by several authors [1–7] with both perturbative and non-perturbative calculations. At present, there is little agreement between the quantitative predictions of such calculations.

A variational method, within the Hamiltonian formalism [8], has been used by Di Leo and Darewych (DLD) [5]. However, because of the apparent complexity of the resulting integral equations, they resorted to additional approximations that are unsatisfactory when the coupling is strong. In this paper we show that an exact decoupling of DLD's integral equations is possible by use of some symmetry properties. This allows a considerable simplification of the problem. For an *s*-wave solution we show in detail that the method gives rise to a one-dimensional integral equation that can be tackled numerically.

Mass renormalization (beyond normal ordering) plays a crucial role, although it is finite. (The infinite mass renormalization of DLD turns out to be an artifact of their other approximations.) The physical mass is significantly reduced from its classical value in the strong-coupling regime. Because of this mass-reduction effect, we find that the occurrence of bound states (i.e., solutions in the 2-particle sector with $E<2m$) is shifted towards the very strong coupling regime, well beyond the reach of any perturbative approximation. However, a relatively small physical mass ($m \approx 0.1-0.5$ TeV) is required as a consequence of the same renormalization effect.

Our paper is organized as follows: the variational method is described in Sec. II. A general prescription for the mass renormalization is then provided and discussed. Section III deals with the delicate aspect of mass renormalization through a variational one-particle trial state calculation analogous to the two-particle trial state used in Sec. II. The existence of Higgs-boson–Higgs-boson bound states is discussed in Sec. IV and the results are compared with those of other authors.

Our discussion here uses the conventional framework that the Higgs theory is an effective theory with a large, but *finite* cutoff Λ [10].

II. VARIATIONAL METHOD

Following DLD, our starting point is the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu h \partial^\mu h - \tfrac{1}{2}m_0^2 h^2 - \frac{1}{3!}\lambda v h^3 - \frac{1}{4!}\lambda h^4, \qquad (1)$$

for a neutral, scalar Higgs field h. For the present we regard m_0, v, and λ as three independent bare parameters; only in Sec. IV shall we need to impose the constraint $m_0^2 = \frac{1}{3}\lambda v^2$ that arises when this model is obtained from a spontaneously broken $\lambda\Phi^4$ theory.

In the Schrödinger representation, at $t=0$, the field is quantized in terms of creation and annihilation operators

$$h(\mathbf{x}) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}} [a_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{x}) + a^\dagger_{\mathbf{k}} \exp(-i\mathbf{k}\cdot\mathbf{x})] \qquad (2)$$

satisfying the usual commutation relations

$$[a_{\mathbf{k}}, a^\dagger{}_{\mathbf{p}}] = \delta^3(\mathbf{k}-\mathbf{p}). \qquad (3)$$

The energy of the single particle states is

$$\omega(\mathbf{k}) = \omega_k = \sqrt{\mathbf{k}^2 + m^2} \qquad (4)$$

where m is the physical mass which may differ from the classical mass m_0. The Hamiltonian H is obtained from Eq. (1) as

$$H = \int d^3k \left(\omega_k - \frac{\delta m^2}{2\omega_k}\right) a^\dagger_{\mathbf{k}} a_{\mathbf{k}} - \int d^3k \frac{\delta m^2}{4\omega_k}(a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} + a_{\mathbf{k}} a_{-\mathbf{k}}) + :\left[\frac{\lambda v}{3!}\int d^3x h^3 + \frac{\lambda}{4!}\int d^3x h^4\right]: \qquad (5)$$

where $\delta m^2 = m^2 - m_0^2$. The Hamiltonian has been normal ordered with respect to the physical mass m.

The preceding discussion follows the conventions of Ref. [5], except that our λ is a factor of 6 larger than theirs, since we use $\lambda/4!$ rather than $\lambda/4$.

The trial two-boson bound state considered by DLD [5] is

$$|\Psi_2\rangle = \int d^3p B(\mathbf{p}) a^\dagger_{\mathbf{p}} a^\dagger_{-\mathbf{p}}|0\rangle + \int d^3p d^3q d^3k G(\mathbf{p},\mathbf{q},\mathbf{k}) a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}} a^\dagger_{\mathbf{k}}|0\rangle \delta^3(\mathbf{p}+\mathbf{q}+\mathbf{k}), \tag{6}$$

with $|0\rangle$ the vacuum annihilated by a. We observe that the function $B(\mathbf{p})$ may be taken to be symmetric without any loss of generality; a general $B(\mathbf{p})$ could always be decomposed into an even and an odd part and the odd part would give no contribution to Eq. (6). This reflects the fact that a bound state of two identical bosons must be even under spatial inversion. Similarly, we must have $G(\mathbf{p},\mathbf{q},\mathbf{k}) = G(-\mathbf{p}, -\mathbf{q}, -\mathbf{k})$. Furthermore, there is no loss of generality in assuming that $G(\mathbf{p},\mathbf{k},\mathbf{q})$ is invariant under any permutation of the three momenta $\mathbf{p}$, $\mathbf{k}$, and $\mathbf{q}$. (Of course, because of the momentum-conserving delta function, the function G really involves only two *independent* momentum arguments.)

The B and G functions are determined from the variational principle

$$\delta\langle\Psi_2|H-E|\Psi_2\rangle = 0 \tag{7}$$

which provides two coupled eigenvalue equations

$$\frac{\delta\langle\Psi_2|H-E|\Psi_2\rangle}{\delta B^\star(\mathbf{k})} = 0 \tag{8}$$

$$\frac{\delta\langle\Psi_2|H-E|\Psi_2\rangle}{\delta G^\star(\mathbf{p},\mathbf{q},\mathbf{k})} = 0. \tag{9}$$

Explicitly, these equations are

$$\left(2\omega_k - \frac{\delta m^2}{\omega_k} - E\right) B(\mathbf{k}) + \frac{\lambda}{64\pi^3\omega_k} \times \int d^3p \frac{B(\mathbf{p})}{\omega_p} + \frac{3\lambda v}{8\pi^{3/2}} \int d^3p \int d^3q\, \delta^3(\mathbf{k}+\mathbf{p}+\mathbf{q}) \times \frac{1}{\sqrt{\omega_k\omega_p\omega_q}} G(\mathbf{k},\mathbf{p},\mathbf{q}) = 0, \tag{10}$$

$$\left[\sum_{i=1}^{3}\left(\omega_i - \frac{\delta m^2}{2\omega_i}\right) - E\right] G(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) + \frac{\lambda v}{(4!)\pi^{3/2}\sqrt{\omega_1\omega_2\omega_3}} \sum_{i=1}^{3} B(\mathbf{k}_i) + \frac{\lambda}{64\pi^3\sqrt{\omega_1\omega_2\omega_3}} \sum_{i=1}^{3} \int d^3p \int d^3q\, \delta^3(\mathbf{k}_i+\mathbf{p}+\mathbf{q}) \times \left(\frac{\omega_i}{\omega_p\omega_q}\right)^{1/2} G(\mathbf{k}_i,\mathbf{p},\mathbf{q}) = 0, \tag{11}$$

where $\omega_i \equiv \omega_{k_i}$. These equations are equivalent to Eqs. (8),(9) of Ref. [5], but their structure appears considerably simpler because we have taken advantage of the symmetry properties mentioned above. We can achieve an exact decoupling of these equations by the following algebraic manipulations. First, we introduce the auxiliary function

$$A(\mathbf{k}) = \omega_k \int d^3p \int d^3q\, \delta^3(\mathbf{k}+\mathbf{p}+\mathbf{q}) \frac{G(\mathbf{k},\mathbf{p},\mathbf{q})}{\sqrt{\omega_k\omega_p\omega_q}} \tag{12}$$

so that the first equation becomes

$$\frac{3\lambda v}{8\pi^{3/2}\omega_k} A(\mathbf{k}) = \left(E - 2\omega_k + \frac{\delta m^2}{\omega_k}\right) B(\mathbf{k}) - \frac{\lambda}{64\pi^3\omega_k} \int \frac{d^3p}{\omega_p} B(\mathbf{p}). \tag{13}$$

Similarly, re-writing its last term in terms of $A(\mathbf{k}_i)$, Eq. (11) becomes

$$G(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \frac{-\lambda}{64\pi^3} \frac{1}{\sqrt{\omega_1\omega_2\omega_3}} \times \frac{\sum_{i=1}^{3}[A(\mathbf{k}_i) + (8\pi^{3/2}/3) v B(\mathbf{k}_i)]}{\left[\sum_{i=1}^{3}(\omega_i - \delta m^2/2\omega_i) - E\right]}. \tag{14}$$

Note that the resulting form of G manifestly respects the symmetry properties invoked earlier. Inserting this equation back into Eq. (12), the second eigenvalue equation Eq. (11) is equivalent to

$$A(\mathbf{k}) + (8\pi^{3/2}/3)\frac{J(\mathbf{k})}{1+J(\mathbf{k})} v B(\mathbf{k}) + \frac{2\omega_k}{1+J(\mathbf{k})} \times \int d^3p \int d^3q\, \delta^3(\mathbf{k}+\mathbf{p}+\mathbf{q}) K(\mathbf{k},\mathbf{p},\mathbf{q})[A(\mathbf{p}) + (8\pi^{3/2}/3) v B(\mathbf{p})] = 0 \tag{15}$$

where the kernel $K(\mathbf{k},\mathbf{p},\mathbf{q})$ is wholly symmetric:

$$K(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=\frac{\lambda}{64\pi^3}\frac{1}{\omega_1\omega_2\omega_3\left[\sum_{i=1}^{3}(\omega_i-\delta m^2/2\omega_i)-E\right]} \tag{16}$$

and

$$J(\mathbf{k})=\omega_k\int d^3pK(\mathbf{k},\mathbf{p},-\mathbf{k}-\mathbf{p}) \tag{17}$$

is a logarithmically divergent integral. We shall regularize it with an energy cutoff, $\sqrt{\mathbf{p}^2+m^2}<\Lambda$.

The eigenvalue equations (13), (15) may now be easily decoupled by replacing A from Eq. (13) into Eq. (15). In this way, we obtain the following integral equation for B:

$$\begin{aligned}\left[E-2\omega_k+\frac{1}{\omega_k}\left(\delta m^2+\lambda v^2\frac{J(\mathbf{k})}{1+J(\mathbf{k})}\right)\right]B(\mathbf{k})\\ =\frac{\lambda}{64\pi^3\omega_k}\frac{1+3J(\mathbf{k})}{1+J(\mathbf{k})}\int\frac{d^3p}{\omega_p}B(\mathbf{p})\\ -\frac{2}{1+J(\mathbf{k})}\int d^3pK(\mathbf{k},\mathbf{p},-\mathbf{k}-\mathbf{p})B(\mathbf{p})\\ \times[\delta m^2+\lambda v^2+\omega_p(E-2\omega_p)].\end{aligned} \tag{18}$$

For an s-wave B function, the angular integration can be performed analytically, yielding a one-dimensional integral equation for B which can be solved by numerical methods. There are two main conceptual problems that must first be dealt with: (i) the mass renormalization parameter δm^2 needs to be determined, and (ii) the integral J is logarithmically divergent and requires regularization, say with an energy cutoff Λ. This last point has to do with the physical interpretation of the whole theory. The current orthodox viewpoint is that the original $\lambda\Phi^4$ theory is only an effective theory, valid up to some finite energy scale Λ that acts as a cutoff. Λ is then another parameter of the theory, in addition to m_0 and λ. We shall adopt this viewpoint in this paper. (For a heterodox viewpoint, see Ref. [10].)

Mass renormalization is crucial since the existence of bound states hinges on the comparison between the energy E and the energy of two free bosons at rest, $2m$. Any attractive self-interaction that tends to bind two particles will also give rise to a reduction of the physical free-particle mass compared to the classical mass m_0. Thus it would not be legitimate to ignore mass renormalization and just impose $m=m_0$.

The form of the left-hand side of Eq. (18) suggests [5] that the desirable mass renormalization is such that the combination

$$\left(\delta m^2+\lambda v^2\frac{J(k)}{1+J(k)}\right) \tag{19}$$

should vanish. Since m^2 should not be k dependent, we define

$$\delta m^2=-\lambda v^2\frac{J(0)}{1+J(0)}. \tag{20}$$

For an infinite cutoff, $J\to\infty$ and we would get

$$m^2=m_0^2-\lambda v^2 \tag{21}$$

which is a *finite* mass renormalization. [In the DLD model, because of their other approximations, the $1+J(0)$ denominator is absent in Eq. (20), so that they found an infinite mass renormalization.]

In the next section, we will show that the above mass renormalization is justified by considering a variational calculation of a one-particle state.

III. SINGLE-PARTICLE MASS RENORMALIZATION

The mass renormalization prescription (20) may be recovered in an analogous self-consistent variational procedure. In this case, the trial state $|\Psi_\mathbf{k}\rangle$ for a single boson with mass m and momentum $\mathbf{k}$ is taken to be

$$|\Psi_\mathbf{k}\rangle=C(\mathbf{k})a_\mathbf{k}^\dagger|0\rangle+\int d^3pD(\mathbf{k},\mathbf{p})a_{\mathbf{p}+\mathbf{k}}^\dagger a_{-\mathbf{p}}^\dagger|0\rangle \tag{22}$$

where $D(\mathbf{k},\mathbf{p})=D(\mathbf{k},-\mathbf{p}-\mathbf{k})$. The variational principle now requires that

$$\delta\langle\Psi_\mathbf{k}|H-E(\mathbf{k})|\Psi_\mathbf{k}\rangle=0 \tag{23}$$

which leads to the coupled eigenvalue equations

$$C(\mathbf{k})\left(\omega_\mathbf{k}-\frac{\delta m^2}{2\omega_\mathbf{k}}-E(\mathbf{k})\right)+\frac{\lambda v}{8\pi^{3/2}}\int\frac{d^3pD(\mathbf{k},\mathbf{p})}{\sqrt{\omega_\mathbf{k}\omega_\mathbf{p}\omega_{\mathbf{k}+\mathbf{p}}}}=0 \tag{24}$$

$$\begin{aligned}D(\mathbf{k},\mathbf{q})\left[\omega_\mathbf{q}+\omega_{\mathbf{k}+\mathbf{q}}-\frac{\delta m^2}{2}\left(\frac{1}{\omega_\mathbf{q}}+\frac{1}{\omega_{\mathbf{k}+\mathbf{q}}}\right)-E(\mathbf{k})\right]\\ +\frac{\lambda}{64\pi^3}\int\frac{d^3pD(\mathbf{k},\mathbf{p})}{\sqrt{\omega_\mathbf{q}\omega_{\mathbf{k}+\mathbf{q}}\omega_\mathbf{p}\omega_{\mathbf{p}+\mathbf{k}}}}\\ +\frac{\lambda v}{16\pi^{3/2}}\frac{C(\mathbf{k})}{\sqrt{\omega_\mathbf{k}\omega_\mathbf{q}\omega_{\mathbf{k}+\mathbf{q}}}}=0.\end{aligned} \tag{25}$$

Eliminating the integral we obtain

$$\frac{D(\mathbf{k},\mathbf{q})}{C(\mathbf{k})}=\frac{1}{8\pi^{3/2}v}\sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{q}}\omega_{\mathbf{k}+\mathbf{q}}}}\times\left[\frac{\omega_{\mathbf{k}}-\frac{\delta m^2}{2\omega_{\mathbf{k}}}-\frac{\lambda v^2}{2\omega_{\mathbf{k}}}-E(\mathbf{k})}{\omega_{\mathbf{q}}+\omega_{\mathbf{k}+\mathbf{q}}-\frac{\delta m^2}{2}\left(\frac{1}{\omega_{\mathbf{q}}}+\frac{1}{\omega_{\mathbf{k}+\mathbf{q}}}\right)-E(\mathbf{k})}\right]. \tag{26}$$

Substituting back in Eq. (24) then for $C\neq 0$ we find

$$E(\mathbf{k})=\omega_{\mathbf{k}}-\frac{1}{2\omega_{\mathbf{k}}}\left[\delta m^2+\lambda v^2\frac{J_0(\mathbf{k})}{1+J_0(\mathbf{k})}\right] \tag{27}$$

where

$$J_0(\mathbf{k})=\frac{\lambda}{64\pi^3}\int^{\Lambda}d^3p\frac{1}{\omega_{\mathbf{p}}\omega_{\mathbf{k}+\mathbf{p}}\left[\omega_{\mathbf{p}}+\omega_{\mathbf{k}+\mathbf{p}}-\frac{\delta m^2}{2}\left(\frac{1}{\omega_{\mathbf{p}}}+\frac{1}{\omega_{\mathbf{k}+\mathbf{p}}}\right)-E(\mathbf{k})\right]}. \tag{28}$$

This has a similar structure and the same ultraviolet behavior as the integral $J(\mathbf{k})$ of Eq. (17). In principle, the presence of the single particle energy $E(\mathbf{k})$ in the denominator requires us to solve Eq. (27) and Eq. (28) self-consistently. However, when $\Lambda\gg m$ (as it should be)

$$J_0(\mathbf{k})\approx J(\mathbf{k})\approx\frac{\lambda}{128\pi^3}\int^{\Lambda}\frac{d^3k}{|\mathbf{k}|^3}\approx\frac{\lambda}{32\pi^2}\ln(\Lambda/m)+\text{finite terms}. \tag{29}$$

In such a limit, we self-consistently obtain $E(\mathbf{k})=\omega_{\mathbf{k}}$ provided we take

$$\delta m^2=-\lambda v^2\frac{J_0}{1+J_0} \tag{30}$$

to be compared to Eq. (20). Once more for $\Lambda\gg m$ we recover the mass renormalization prescription (21). We stress that, for a finite energy cutoff Λ, Eq. (30) is analogous but not equivalent to Eq. (20) since $J_0\neq J$. Thus a consistent solution of the full integral equation (18) requires the use of the mass condition (20) as discussed in the next section.

Full consistency would also require that when $E(\mathbf{k})=\omega_{\mathbf{k}}$ the function D should vanish. In fact, since $\omega_{\mathbf{k}}$ is the energy of a single particle state $|\mathbf{k}\rangle=a^{\dagger}_{\mathbf{k}}|0\rangle$ with mass m, the state $|\Psi_{\mathbf{k}}\rangle$ can have a mass m and a Lorentz-covariant dispersion relation only when $|\Psi_{\mathbf{k}}\rangle=|\mathbf{k}\rangle$. This requirement is not entirely trivial: the mass renormalization prescription (30) is just what we need in order to guarantee that the single particle state $|\mathbf{k}\rangle$ effectively is the lower energy one-particle state for the full interacting Hamiltonian (5). Inserting Eq. (27) and (30) into Eq. (26), we find that the ratio $D/C\to 0$ for $\Lambda\to\infty$ as we expected. In other words that means we find a quantum-field renormalization constant $Z=1$.

It is instructive examining the same result from the point of view of standard perturbation theory. If there were some ''rule'' forbidding the existence of states with more than two particles in the Fock space, then the variational trial state (22) would lead to an exact result. The same result should be achievable by perturbation theory provided that we sum all Feynman diagrams whose intermediate states do not contain more than two particles. However, in the self-consistent variational procedure the mass of the single particle state $|\mathbf{k}\rangle$ is supposed to be the true physical mass $m\neq m_0$. In the language of perturbation theory this is equivalent to associate a renormalized propagator with each internal line of Feynman diagrams. The easiest way to do that is by re-writing the Lagrangian (1) as

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2 \tag{31}$$

where $\mathcal{L}_0$ is the zeroth-order non-interacting part:

$$\mathcal{L}_0=-\tfrac{1}{2}\partial_\mu h_R\partial^\mu h_R-\tfrac{1}{2}m^2h_R^2. \tag{32}$$

$\mathcal{L}_1$ is an interaction part contributing at the tree level,

$$\mathcal{L}_1=-\tfrac{1}{2}(Z-1)\partial_\mu h_R\partial^\mu h_R-\tfrac{1}{2}(Z-1)m^2h_R^2+\tfrac{1}{2}Z\,\delta m^2h_R^2, \tag{33}$$

and $\mathcal{L}_2$ is the interaction,

$$\mathcal{L}_2=-\frac{1}{3!}\lambda v Z^{3/2}h_R^3-\frac{1}{4!}Z^2\lambda h_R^4. \tag{34}$$

Here h_R is a renormalized field, $h_R=h/\sqrt{Z}$. Imposing that the renormalized propagator has a pole at $p^2=-m^2$ with unit residue gives two conditions [11]

$$Z\,\delta m^2=-\Pi^{\star}_{loop}(-m^2) \tag{35}$$

and

$$Z = 1 + \frac{d}{dp^2}\Pi^{\star}_{loop}(p^2)|_{-m^2} \quad (36)$$

where $p^2 = \mathbf{p}^2 - \omega_{\mathbf{p}}^2$, and $i(2\pi)^4\Pi^{\star}_{loop}(p^2)$ is the sum of all one-particle-irreducible diagrams containing loops. Such diagrams can only arise from the Lagrangian part $\mathcal{L}_2$. In our reduced Fock space the only diagrams contributing are the bubble-chain diagrams reported in Fig. 1. These are all naively divergent but, if a regularization prescription allows their resummation, their infinite sum yields a finite contribution

$$\Pi^{\star}_{loop}(p^2) = \Pi^{\star}_{1\text{-}loop} + \Pi^{\star}_{1\text{-}loop}(p^2)\cdot\left[-\frac{1}{2}Z^2\lambda I(p^2)\right] + \Pi^{\star}_{1\text{-}loop}(p^2)\cdot\left[-\frac{1}{2}Z^2\lambda I(p^2)\right]^2 + \cdots \quad (37)$$

where the one-loop term is

$$\Pi^{\star}_{1\text{-}loop}(p^2) = \frac{1}{2}(Z^3\lambda^2 v^2)I(p^2) \quad (38)$$

and $I(p^2)$ is the divergent integral:

$$I(p^2) = \int \frac{d^4q}{i(2\pi)^4}\frac{1}{(q^2+m^2-i\epsilon)[(p+q)^2+m^2-i\epsilon]}. \quad (39)$$

The infinite series (37) can be exactly summed up:

$$\Pi^{\star}_{loop}(p^2) = (Z\lambda v^2)\left[\frac{\frac{1}{2}\lambda Z^2 I(p^2)}{1+\frac{1}{2}\lambda Z^2 I(p^2)}\right]. \quad (40)$$

In order to maintain Lorentz covariance, the integral (39) can be evaluated by dimensional regularization. By use of the Feynman formula and Wick rotation, in a d-dimensional Euclidean space the integral reads

$$I(p^2) = \frac{\pi^{d/2}}{(2\pi)^4}\Gamma(2-d/2)\int_0^1[m^2+p^2x(1-x)]^{d/2-2}dx. \quad (41)$$

For $d = 4+\epsilon$ and $\epsilon \to 0$ we obtain

$$I(p^2) = -\frac{1}{8\pi^2\epsilon} - \frac{1}{16\pi^2}\times\left[\gamma + \ln\pi + \int_0^1 dx\,\ln[m^2+p^2x(1-x)]\right] + \mathcal{O}(\epsilon). \quad (42)$$

Insertion into Eq. (40) gives

$$\frac{d}{dp^2}\Pi^{\star}_{loop}(p^2)|_{-m^2} \sim \epsilon^2 \quad (43)$$

and from Eq. (36) then

$$Z = 1 + \mathcal{O}(\epsilon^2) \quad (44)$$

while from Eq. (35) the mass renormalization parameter reads

$$\delta m^2 = -\lambda v^2\left[\frac{\lambda/(16\pi^2\epsilon)}{\lambda/(16\pi^2\epsilon)-1}\right]. \quad (45)$$

Thus, in physical $d=4$ space, we recover for $\epsilon \to 0$ the mass renormalization prescription of Eq. (21) and $Z=1$.

IV. SEARCH FOR HIGGS-BOSON–HIGGS-BOSON BOUND STATES

In the previous sections we described the variational method and discussed its internal consistency from a general point of view. So far the three parameters m_0^2, v and λ have been viewed as independent and we have not specialized to any particular physical problem. We now wish to use the method described in Sec. II to search for Higgs-boson–Higgs-boson bound states in the Higgs sector of the electroweak theory.

The scalar sector of the electroweak theory has the form

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi - \frac{\lambda}{4!}(\Phi^2 - v^2)^2 + \text{const.} \quad (46)$$

Defining the ''Higgs'' field h by $\Phi = v + h$, we obtain the original Lagrangian (1) with the parameters related by

$$m_0^2 = \frac{1}{3}\lambda v^2. \quad (47)$$

It can be shown that the same relation holds for v being the minimum of the Gaussian effective potential [9]. In the standard interpretation, we also have a large but finite cutoff Λ. The theory is approximately Lorentz invariant for energies small compared to a finite energy cutoff Λ. The vacuum value v is fixed empirically in terms of the Fermi constant G_F:

$$2v^2 = \frac{\sqrt{2}}{G_F}. \quad (48)$$

Thus the bare mass m_0 is proportional to the square root of the coupling λ. The physical model is entirely described by

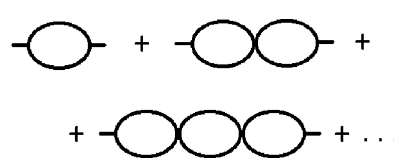

FIG. 1. Bubble-chain diagrams contributing to $\Pi^{\star}_{loop}$ in Eq. (37).

two independent energy scales: the energy cutoff Λ and the bare mass m_0 which also fixes the coupling strength through Eq. (47).

For a large cutoff Λ, according to Eq. (21), we expect a mass correction $\delta m^2 \approx -\lambda v^2$ which overcomes the tree-level mass (47). Since m^2 is positive definite, we expect that something should prevent it from becoming negative. In fact the general discussion of Sec. II must be modified when the physical mass m approaches zero. At the point $m \to 0$, $\Lambda \to \infty$ the integral $J(0)$ is not analytical, and some extra care is required in handling the two limits. As previously discussed $J(0)$ diverges logarithmically according to Eq. (29) for any finite m, while it vanishes linearly for $m \to 0$ at any fixed cutoff Λ. Thus for any large but finite Λ the coupled equations (17) and (20) must be solved together yielding a real cutoff dependent mass $m(\Lambda)$. Of course at any finite cutoff such equations maintain a k dependence since the theory is not Lorentz invariant. We define m as the $\mathbf{k}=0$ value corresponding to the energy required to create a boson at rest. For $\mathbf{k}=0$ a generic scattering solution of the integral equation (18) has $E=2m$ where m is determined from Eq. (20), by insertion of Eq. (47),

$$m^2 = m_0^2\left[\frac{1-2J(0)}{1+J(0)}\right], \tag{49}$$

while $J(0)$ is the integral (17) of the kernel (16) evaluated at $\mathbf{k}_1=0$ and at $E=2m$:

$$J(0) = \frac{\lambda}{32\pi^2}\int_1^g \sqrt{\frac{x^2-1}{x^2-\alpha x-\beta}}\, dx \tag{50}$$

where $g=\Lambda/m$, $\alpha=(3-m_0^2/m^2)/4$ and $\beta=(1-m_0^2/m^2)/2$.

The two coupled equations (49) and (50) give the physical mass m. The integral in Eq. (50) can be evaluated analytically:

$$J(0) = \frac{3m_0^2}{32\pi^2 v^2} f(g), \tag{51}$$

where

$$\begin{aligned} f(g) = {} & \ln(g+\sqrt{g^2-1}) \\ & + \frac{1}{2}\sum_{\pm} \frac{\sqrt{\gamma_\pm^2-1}}{\pm\sqrt{\Delta}} \ln(1-g\gamma_\pm+\sqrt{g^2-1}\sqrt{\gamma_\pm^2-1}) \\ & - \frac{1}{2}\sum_{\pm} \frac{\sqrt{\gamma_\pm^2-1}}{\pm\sqrt{\Delta}} \ln(1-g\gamma_\pm-\sqrt{g^2-1}\sqrt{\gamma_\pm^2-1}), \end{aligned} \tag{52}$$

and $\gamma_\pm=(\alpha\pm\sqrt{\Delta})/2$ and $\Delta=\alpha^2+4\beta$.

A numerical solution for m as a function of the coupling parameter m_0 is reported in Fig. 2 for several cutoff values. In the weak-coupling limit, for m_0 smaller than 0.3 TeV, we recover the perturbative solution $m \approx m_0$ which holds up to a quite huge Λ. That is equivalent to neglecting $J(0)$ altogether in Eq. (49). For larger couplings the solution deviates from the perturbative regime and the physical mass m is heavily reduced and strongly dependent on the cutoff choice. Notably in the range 1 TeV$<m_0<$2 TeV, where several authors find Higgs-boson–Higgs-boson bound states, the physical mass is spread over a large energy range going from $m=m_0$ for $\Lambda=m_0$ to $m \approx m_0/100$ for $\Lambda=35$ TeV. However, the upper bound $m=m_0$ is only reached for an unphysical cutoff equal to the mass, which makes the integral J vanish. A crossover is observed at a critical $\Lambda_c=3.05$ TeV from a monotonic increasing behavior of m versus m_0, to a non-monotonic behavior with m rising to a maximum and then decreasing for larger m_0. The maximum value of m never exceeds the critical value $m_c=1.1$ TeV (which is reached at a coupling $m_0=4$ TeV) for any choice of coupling and cutoff. Thus, for any $\Lambda>3.05$ TeV and any m_0, we always find a physical mass $m<1.1$ TeV.

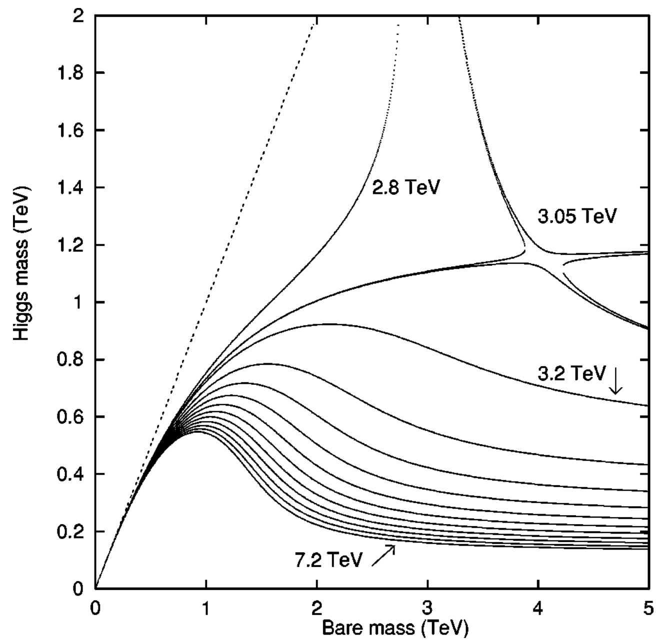

FIG. 2. The physical Higgs boson mass m versus the bare mass m_0 (which fixes the coupling strength), for several choices of the energy cutoff $\Lambda=2.8$, 3.2, 3.6, 4.0, 4.4, 4.8, 5.2, 5.6, 6.0, 6.4, 6.8 and 7.2 TeV. Data are also reported for the crossover point from monotonic to non-monotonic behavior occurring at $\Lambda_c=3.05$ TeV. The dashed line represents the $m=m_0$ approximation which only holds in the weak-coupling regime $m_0<0.3$ TeV.

Moreover, in the strong coupling regime the physical mass becomes very small: in such strong-coupling limit a simple analytical solution may be found by requiring that $m \ll m_0$ and expanding both equations (49) and (50) in powers of m^2/m_0^2. By use of the analytical expression (52), the integral (50) may be written as

$$J(0) = \frac{3m\Lambda}{8\pi^2 v^2} + \mathcal{O}(m^2/m_0^2) \tag{53}$$

which to first order in m does not depend on the coupling m_0. Equation (49) may be inverted and to the same order in m^2 reads

$$J(0)=\frac{1}{2}+\mathcal{O}(m^2/m_0^2). \tag{54}$$

Eliminating $J(0)$ yields

$$m\approx\frac{4}{3}\pi^2\frac{v^2}{\Lambda} \tag{55}$$

which is consistent with our assumption that $m\ll m_0$ provided that $m_0\gg 4\pi^2 v^2/(3\Lambda)$ or explicitly $m_0\Lambda\gg 0.8\ \text{TeV}^2$. When such condition is satisfied the physical mass does not depend on the coupling and tends to a finite limit proportional to Λ^{-1}. This behavior is evident in Fig. 2.

For completeness we should mention that whenever $m_0>\Lambda$ a second larger solution for the physical mass m comes out from the coupled equations (49), (50), but such solutions probably have no physical meaning. We remark that the range $m_0>\Lambda$ could be of physical interest since the cutoff Λ should be compared to the physical mass m which is generally significantly smaller than the coupling parameter m_0.

Returning to the bound-state problem, let us insert the numerical solution of the coupled equations (49), (51) into the integral equation (18). Neglecting the slight $\mathbf{k}$ dependence of $J(\mathbf{k})\approx J(0)$ (which is the only approximation we are making apart from the choice of the trial state) we obtain the following integral equation:

$$(2\omega_k-E)B(\mathbf{k})=-\int d^3p\,\mathcal{K}(\mathbf{k},\mathbf{p},-\mathbf{k}-\mathbf{p})B(\mathbf{p}), \tag{56}$$

where the kernel $\mathcal{K}$ is defined as

$$\mathcal{K}(\mathbf{k},\mathbf{p},\mathbf{q})=\frac{1}{64\pi^3\omega_k}\left(\frac{2m_0^2+m^2}{v^2}\right)\left\{\left(\frac{5m_0^2-2m^2}{2m_0^2+m^2}\right)\frac{1}{\omega_p}-\frac{2}{\omega_p\omega_q}\frac{m^2+2m_0^2+\omega_p(E-2\omega_p)}{\left[\omega_k+\omega_p+\omega_q+\dfrac{m_0^2-m^2}{2}\left(\dfrac{1}{\omega_k}+\dfrac{1}{\omega_p}+\dfrac{1}{\omega_q}\right)-E\right]}\right\}. \tag{57}$$

We notice that the second term inside the brackets contains an attractive part plus an energy dependent part proportional to $(E-2\omega_p)$ which is always repulsive if $E<2m$, and thus weakens the bonding of any bound state. It is instructive to see how the approximate integral equation of DLD [5] can be recovered from our almost exact treatment of the variational method. That can be done by taking the perturbative limit $[J(0)=0,\ m=m_0]$ and by neglecting the energy dependent repulsive part proportional to $(E-2\omega_p)$ in the second term of the kernel according to its approximation $E\approx 2\omega_p$. Moreover, since they also take $E\approx\omega_p+\omega_k\approx\omega_q+\omega_k$, the denominator of the second term is approximated as

$$\frac{1}{\omega_k+\omega_p+\omega_q-E}\approx\frac{1}{\omega_p}\approx\frac{1}{\omega_q}\approx\frac{1}{2}\left(\frac{1}{\omega_p}+\frac{1}{\omega_q}\right), \tag{58}$$

yielding the approximation

$$\mathcal{K}(\mathbf{k},\mathbf{p},\mathbf{q})\approx\frac{\lambda}{64\pi^3}\frac{3m^2}{\omega_k\omega_p}\left[\frac{1}{3m^2}-\frac{1}{\omega_q^2}-\frac{1}{\omega_q\omega_p}\right] \tag{59}$$

which is Eq. (16) of Ref. [5]. This approximation requires $m\approx m_0$, which is valid only in the region $m_0<0.3$ TeV, according to our numerical results in Fig. 2. DLD found bound-state solutions with the kernel (59), but only for larger couplings $m_0>0.9$ TeV; this is well beyond the perturbative regime [5,12,13] and well beyond the region of validity of their approximations.

To address the problem of the existence of bound state beyond the perturbative regime, we must use the full integral equation (56). The most interesting case is an s-wave solution, and in that case the integration over angles may be carried out exactly, yielding a one-dimensional integral equation. We change the integration variables according to

$$\int d^3p\cdots=\frac{2\pi}{\sqrt{\omega_k^2-m^2}}\int_m^\infty\omega_p d\omega_p\int_{\omega_-}^{\omega_+}\omega_q d\omega_q\cdots, \tag{60}$$

where

$$\omega_\pm=\sqrt{\omega_k^2+\omega_p^2-m^2\pm 2\sqrt{(\omega_k^2-m^2)(\omega_p^2-m^2)}}. \tag{61}$$

Then integrating over ω_q gives

$$(2\omega_k - E)B(\mathbf{k}) = \int_m^{\Lambda} \mathcal{F}(\omega_k, \omega_p) B(\mathbf{p}) d\omega_p, \tag{62}$$

with

$$\mathcal{F}(\omega_k,\omega_p) = -\frac{2m_0^2+m^2}{16\pi^2 v^2 \omega_k}\left\{ \left(\frac{5m_0^2-2m^2}{2m_0^2+m^2}\right)\sqrt{\omega_p^2-m^2} - \frac{2m_0^2+m^2+\omega_p(E-2\omega_p)}{\sqrt{\omega_k^2-m^2}} \times\left[\frac{\Omega_+}{\Omega_+-\Omega_-}\ln\left(\frac{\omega_+-\Omega_+}{\omega_--\Omega_+}\right) - \frac{\Omega_-}{\Omega_+-\Omega_-}\ln\left(\frac{\omega_+-\Omega_-}{\omega_--\Omega_-}\right)\right]\right\} \tag{63}$$

where $\Omega_\pm$ are the poles of the kernel $\mathcal{K}$ in the variable ω_q:

$$\Omega_\pm = \frac{E-\omega_k-\omega_p}{2} - \frac{m_0^2-m^2}{4}\left(\frac{1}{\omega_k}+\frac{1}{\omega_p}\right) \pm \frac{1}{2}\sqrt{\left[E-\omega_k-\omega_p - \frac{m_0^2-m^2}{2}\left(\frac{1}{\omega_k}+\frac{1}{\omega_p}\right)\right]^2 - 2m_0^2+2m^2}. \tag{64}$$

The one-dimensional integral equation (62) may be solved numerically by standard matrix techniques. [One must first choose the parameters m_0 and Λ, and find the corresponding physical mass m from the coupled equations (49), (51).] We can describe three different scenarios:

(i) $\Lambda < \Lambda_c = 3.05$ TeV (small cutoff). Since m is a monotonic increasing function of the coupling strength m_0, both m and m_0 must be small compared to Λ. In this weak-coupling limit there are no bound-state solutions and the lower eigenvalue of Eq. (62) is the free particle energy $E = 2m$.

(ii) $\Lambda > \Lambda_c$, $m_0 < 2$ TeV (large cutoff and moderately strong coupling). Beyond the critical value $\Lambda_c = 3.05$ TeV the Higgs boson mass m is not a monotonic increasing function of the coupling strength m_0 (see Fig. 2). This strength can be large since m is bounded and never comparable to Λ. Beyond the weak-coupling limit, where there are no bound-state solutions, an intermediate range can be described for $m_0 \approx 1-2$ TeV. In this range nonperturbative effects are evident since m is a *decreasing* function of the bare mass m_0 (as shown in Fig. 2). In this range bound state solutions have been found by several authors [1,2,4,5]. However, the strong reduction of the physical mass m, in comparison with the bare mass m_0, rules out the existence of bound states with $E < 2m$ in this regime.

(iii) $\Lambda > \Lambda_c$, $m_0 > 2$ TeV (very strong coupling). For very large m_0 the renormalized Higgs boson mass m saturates at the finite value given by Eq. (55) (see also Fig. 2). A further increase of the coupling strength allows the occurrence of bound-state solutions whose precise onset depends on the chosen energy cutoff Λ. In Fig. 3 we show the binding energy $(E-2m)$, in units of $2m$, for a typical cutoff $\Lambda = 4$ TeV, just above Λ_c. An eigenvalue E smaller than $2m$ appears at $m_0 = 2.35$ TeV, and the binding energy reaches the bootstrap point $(E-2m)/(2m) = -0.5$ at $m_0 = 3.5$ TeV.

Despite the huge couplings required for binding, the corresponding physical mass is relatively small compared to that found in previous works [1–5]. Figure 4 reports the binding energy versus m (physical mass) for $\Lambda = 4$ TeV. The onset of the bound-state solution is at $m = 519$ GeV, while the mass bootstrap point is reached at $m = 386$ GeV. We notice that the binding energy now *increases* with decreasing m. Moreover, according to Eq. (55), an even smaller mass is required for larger choices of the energy cut-off Λ.

At the light of our study a Higgs-boson–Higgs-boson bound state would be conceivable for $m \approx 100-500$ GeV provided that the coupling is very strong, $m_0 \approx 2$ TeV. Of course the present toy model neglects the interactions with the longitudinal components of W and Z fields, and the nu-

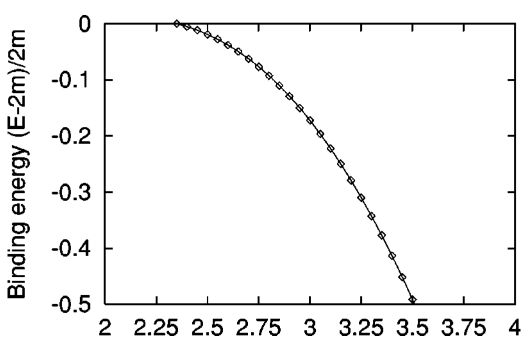

FIG. 3. Binding energy $E-2m$ in units of $2m$ versus bare mass m_0 (coupling strength) for a cutoff $\Lambda = 4$ TeV.

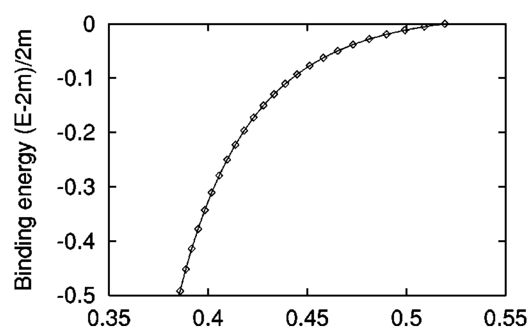

FIG. 4. Binding energy $E-2m$ in units of $2m$ versus physical Higgs boson mass m for a cutoff $\Lambda = 4$ TeV.

merical results might not apply to real bound states of the standard model. We stress the role played by mass renormalization in determining both the shift of bound states towards higher coupling strengths and the corresponding reduction of the physical mass required for bonding. Most of the previous calculations should be revised at the light of the present result in order to establish if mass renormalization has been correctly addressed. We just mention Rupp's [4] Bethe-Salpeter approach where the chosen subtraction point gives $m = m_0$ at any coupling.

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